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CERTAIN INVARIANTS OF A QUADRANGLE BY PROJECTIVE TRANSFORMATION.

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A geometrical configuration depending on m independent parameters has $m - r$ independent invariants by an r parameter Lie group of point transformations; these $m - r$ invariants are found by the integration of the complete system of simultaneous partial differential equations

$$X_1 f = 0 \ , \ \ldots , \ X_r f = 0 ,$$

where the $X_i f$ are the r independent infinitesimal transformations which generate the r parameter group.

The object of this note is to determine the invariants of a plane quadrangle, i. e. of a system of four points in the plane, by several projective groups of the plane.

1. The general projective group of the plane is an eight parameter group generated by the following eight independent infinitesimal transformations :

$$\boxed{p \quad q \quad xp \quad yq \quad xq \quad yp \quad x^2p + xyq \quad xyp + y^2q}$$

where

$$p \equiv \frac{\partial f}{\partial x} , \ q \equiv \frac{\partial f}{y} .$$

The quadrangle is a configuration depending on eight independent parameters, namely, the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) ; accordingly the quadrangle has no invariants by the general projective group. If, however, the quadrangle degenerates into a system of four collinear points, the anharmonic ratio of the four points is a known invariant by the general projective group. Hence, for this case, if we construct the determinant whose vanishing expresses the condition that the simultaneous system

$$\begin{gathered}
\overset{4}{\underset{1}{\Sigma}}{}_i \frac{\partial \varphi}{\partial x_i} = 0 , \quad \overset{4}{\underset{1}{\Sigma}}{}_i \frac{\partial \varphi}{\partial y_i} = 0 , \quad \overset{4}{\underset{1}{\Sigma}}{}_i x_i \frac{\partial \varphi}{\partial x_i} = 0 , \\
\overset{4}{\underset{1}{\Sigma}}{}_i y_i \frac{\partial \varphi}{\partial y_i} = 0 , \quad \overset{4}{\underset{1}{\Sigma}}{}_i x_i \frac{\partial \varphi}{\partial y_i} = 0 , \quad \overset{4}{\underset{1}{\Sigma}}{}_i y_i \frac{\partial \varphi}{\partial x_i} = 0 , \qquad (1) \\
\overset{4}{\underset{1}{\Sigma}}{}_i \left[x_i^2 \frac{\partial \varphi}{\partial x_i} + x_i y_i \frac{\partial \varphi}{\partial y_i} \right] = 0 , \quad \overset{4}{\underset{1}{\Sigma}}{}_i \left[x_i y_i \frac{\partial \varphi}{\partial x_i} + y_i^2 \frac{\partial \varphi}{\partial y_i} \right] = 0 ,
\end{gathered}$$

have a solution, we find the following theorem :

If the four points $(x_1, y_1,)$, (x_2, y_2), (x_3, y_3), (x_4, y_4) *are collinear the determinant*

$$D \equiv \begin{vmatrix} x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{vmatrix} \tag{2}$$

vanishes.

2. There is no seven parameter projective group in the plane. There are two six parameter projective groups. The first of these is the general linear group

$$\boxed{p \quad q \quad xp \quad yp \quad xq \quad yq}\,. \tag{a}$$

The invariants sought are the solutions of the complete system of partial differential equations

$$\sum_{1}^{4}{}_i \frac{\partial\varphi}{\partial x_i} = \sum_{1}^{4}{}_i \frac{\partial\varphi}{\partial y_i} = \sum_{1}^{4}{}_i x_i \frac{\partial\varphi}{\partial x_i} = \sum_{1}^{4}{}_i y_i \frac{\partial\varphi}{\partial x_i} = \sum_{1}^{4}{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_{1}^{4}{}_i y_i \frac{\partial\varphi}{\partial y_i} = 0\,. \tag{3}$$

The first and second of these equations assert that the invariant functions $\varphi(x_i, y_i)$ depend on the differences

$$u_1 = x_2 - x_1\,, \quad u_2 = x_3 - x_1\,, \quad u_3 = x_4 - x_1\,;$$

$$v_1 = y_2 - y_1\,, \quad v_2 = y_3 - y_1\,, \quad v_3 = y_4 - y_1\,;$$

so that the third and sixth may be written, respectively,

$$\sum_{1}^{3}{}_i u_i \frac{\partial\varphi}{\partial u_i} = 0\,, \quad \sum_{1}^{3}{}_i v_i \frac{\partial\varphi}{\partial v_i} = 0\,; \tag{4}$$

and the fourth and fifth become, respectively,

$$\sum_{1}^{3}{}_i v_i \frac{\partial\varphi}{\partial u_i} = 0\,, \quad \sum_{1}^{3}{}_i u_i \frac{\partial\varphi}{\partial v_i} = 0\,. \tag{5}$$

The equations (4) are equivalent to the simultaneous systems

$$\frac{du_1}{u_1} = \frac{du_2}{u_2} = \frac{du_3}{u_3}\,; \quad \frac{dv_1}{v_1} = \frac{dv_2}{v_2} = \frac{dv_3}{v_3}\,,$$

which show that φ is a function only of the variables

$$s_1 = \frac{u_2}{u_1}\,, \quad s_2 = \frac{u_3}{u_1}\,, \quad t_1 = \frac{v_2}{v_1}\,, \quad t_2 = \frac{v_3}{v_1}\,.$$

If these be introduced as new variables in equations (5), the resulting equations show that the invariant function is a function of $(s_1 - t_1)/(s_2 - t_2)$.

Developing $(s_1 - t_1)/(s_2 - t_2)$ and its analogous forms we find the three invariant equations

$$| u_1, v_2 |/| u_1, v_3 | = c_1\,, \quad | u_1, v_2 |/| u_2, v_3 | = c_2\,, \quad | u_1, v_3 |/| u_2, v_3 | = c_3\,,$$

only two of which are independent,* as was to be expected since an invariant function of eight independent variables by a six parameter group was sought. In order to interpret these invariants geometrically we develop c_1, c_2, and c_3 still further. In fact

$$c_1 = | u_1, v_2 |/| u_1, v_3 | = \frac{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}{(x_4 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_4 - y_1)} = \frac{\triangle 123}{\triangle 124}\,;$$

similarly

$$c_2 = \frac{\triangle 132}{\triangle 134}\,, \quad c_3 = \frac{\triangle 142}{\triangle 143}\,.$$

Hence the theorem :

If the quadrangle (1234) *be transformed by the* Lie *group*

$$\boxed{p \quad q \quad xp \quad yp \quad xq \quad yq}$$

the ratios of the triangle (123) *to the triangles* (124) *and* (134) *remain unchanged ; i. e. the area of the quadrangle is changed in a constant ratio.*

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* That there are not more than two independent solutions of the original complete system (3) is seen from the fact that all sixth order determinants of the matrix

$$\left|\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \end{array}\right|$$

do not vanish identically.

It is of course obvious that any function of c_1, c_2, and c_3 is invariant by the transformations of the group. The result of this article might have been anticipated. The group is the so-called general linear group of the plane whose characteristic property is that by it all areas are changed in the same ratio.

3. The only other six parameter projective group in the plane is the group

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q} \,. \tag{β}$$

The corresponding complete system to be solved is

$$\begin{gathered}\sum_1^4{}_i x_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = \sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial y_i} = 0\,, \\ \sum_1^4{}_i \left[x_i^2 \frac{\partial\varphi}{\partial x_i} + x_i y_i \frac{\partial\varphi}{\partial y_i}\right] = \sum_1^4{}_i \left[x_i y_i \frac{\partial\varphi}{\partial x_i} + y_i^2 \frac{\partial\varphi}{\partial y_i}\right] = 0\,.\end{gathered} \tag{6}$$

The first and fourth of these equations give φ as a function of the variables

$$\bar u_1 = \frac{x_2}{x_1}\,, \quad \bar u_2 = \frac{x_3}{x_1}\,, \quad \bar u_3 = \frac{x_4}{x_1}\,; \quad \bar v_1 = \frac{y_2}{y_1}\,, \quad \bar v_2 = \frac{y_3}{y_1}\,, \quad \bar v_3 = \frac{y_4}{y_1}\,.$$

These introduced as new variables reduce the second and third equations respectively to

$$\sum_1^3{}_i (\bar u_i - \bar v_i) \frac{\partial\varphi}{\partial \bar u_i} = 0\,, \quad \sum_1^3{}_i (\bar u_i - \bar v_i) \frac{\partial\varphi}{\partial \bar v_i} = 0\,; \tag{7}$$

and the fifth and sixth respectively to

$$\begin{gathered}\sum_1^3{}_i \left\{ \bar u_i (\bar u_i - 1) \frac{\partial\varphi}{\partial \bar u_i} + \bar v_i (\bar u_i - 1) \frac{\partial\varphi}{\partial \bar v_i} \right\} = 0\,, \\ \sum_1^3{}_i \left\{ \bar u_i (\bar v_i - 1) \frac{\partial\varphi}{\partial \bar u_i} + \bar v_i (\bar v_i - 1) \frac{\partial\varphi}{\partial \bar v_i} \right\} = 0\,.\end{gathered} \tag{8}$$

The equations (8) are satisfied if φ is a function of the variables

$$r_1 = \frac{\bar u_1}{\bar v_1}\,, \quad r_2 = \frac{\bar u_2}{\bar v_2}\,, \quad r_3 = \frac{\bar u_3}{\bar v_3}\,;$$

these introduced in equations (7) reduce them to the forms respectively

$$\sum_1^3{}_i (r_i - 1) \frac{\partial\varphi}{\partial r_i} = 0\,, \quad \sum_1^3{}_i r_i (r_i - 1) \frac{\partial\varphi}{\partial r_i} = 0\,, \tag{9}$$

the first of which admits of the solutions

$$\frac{r_2 - 1}{r_1 - 1} = s_1 , \quad \frac{r_3 - 1}{r_1 - 1} = s_2 ,$$

which substituted in the second produce the general solution

$$\frac{s_2 - 1}{s_1 - 1} = \text{constant} .$$

Expanding this last expression there results

$$\frac{s_2 - 1}{s_1 - 1} = \frac{r_3 - r_1}{r_2 - r_1} = \frac{\bar{u}_2 \, | \bar{u}_3, \bar{v}_1 |}{\bar{v}_2 \, | \bar{u}_2, \bar{v}_1 |} = \frac{y_3 \, | x_4, y_2 |}{y_4 \, | x_3, y_2 |} = \frac{y_3 \triangle 024}{y_4 \triangle 023} ,$$

where zero is the origin. Hence the theorem

If a quadrangle (1234) *is subjected to the transformations of the* Lie *group*

$$\boxed{\; xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q \;}$$

the ratio of the triangles (023) *and* (024) *varies as the ratio of the ordinates* y_3 *and* y_4*; similarly the ratio of the triangles* (012) *and* (013) *varies as the ratio of the abscissas* x_2 *and* x_3 *; these two factors of proportionality are the independent invariants.*

That there are no more than two independent invariants may be verified by constructing the test matrix as in the foot-note to the preceding section and observing that all sixth order determinants do not vanish.

4. There are three types of five parameter projective groups of the plane. We shall take space to find the invariants of the quadrangle by two of them because of the beauty of the geometrical results. The group

$$\boxed{\; p \quad q \quad xq \quad xp - yq \quad yp \;} \qquad (\lambda)$$

yields the complete system

$$\sum_1^4{}_i \frac{\partial \varphi}{\partial x_i} = \sum_1^4{}_i \frac{\partial \varphi}{\partial y_i} = \sum_1^4{}_i x_i \frac{\partial \varphi}{\partial y_i} = \sum_1^4{}_i \left[x_i \frac{\partial \varphi}{\partial x_i} - y_i \frac{\partial \varphi}{\partial y_i} \right] = \sum_1^4{}_i y_i \frac{\partial \varphi}{\partial x_i} = 0 . \qquad (10)$$

This system of five equations in eight variables has three independent solutions but no more as its matrix shows.

The first two equations make φ a function of the differences

$$u_1 = x_2 - x_1, \quad u_2 = x_3 - x_1 , \quad u_3 = x_4 - x_1 ;$$

$$v_1 = y_2 - y_1, \quad v_2 = y_3 - y_1 , \quad v_3 = y_4 - y_1 ;$$

and these substituted in the last three equations lead to the three independent solutions

$$| u_1, v_2 | \equiv 2\triangle 123 = c_1, \quad | u_1, v_3 | \equiv 2\triangle 124 = c_2, \quad | u_2, v_3 | \equiv 2\triangle 134 = c_3 ;$$

whence the theorem

The area of the quadrangle (1234) *is left invariant by the* LIE *group*

$$\boxed{p \quad q \quad xq \quad xp - yq \quad yp} .$$

This invariant might have been anticipated had it been observed that the group is the so-called special linear group whose peculiar property is that of leaving all areas invariant.

5. Consider finally the five parameter group

$$\boxed{xq \quad xp - yq \quad yp \quad x^2p + xyq \quad xyp + y^2q} \qquad (\mu)$$

the simultaneous system to be satisfied by the invariant functions sought is made up of the following equations:

$$\sum_1^4{}_i x_i \frac{\partial\varphi}{\partial y_i} = \sum_1^4{}_i \left[x_i \frac{\partial\varphi}{\partial x_i} - y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i y_i \frac{\partial\varphi}{\partial x_i} = 0 ,$$

$$\sum_1^4{}_i \left[x_i^2 \frac{\partial\varphi}{\partial x_i} + x_i y_i \frac{\partial\varphi}{\partial y_i} \right] = \sum_1^4{}_i \left[x_i y_i \frac{\partial\varphi}{\partial x} + y_i^2 \frac{\partial\varphi}{\partial y_i} \right] = 0 .$$

The first and third demand the φ be a function of the variables

$$u_1 = \frac{x_2}{x_1}, \quad \bar{u}_2 = \frac{x_3}{x_1}, \quad \bar{u}_3 = \frac{x_4}{x_1}; \quad \bar{v}_1 = \frac{y_2}{y_1}, \quad \bar{v}_2 = \frac{y_3}{y_1}, \quad \bar{v}_3 = \frac{y_4}{y_1} .$$

These are found to satisfy the second equation identically and to reduce the fourth and fifth respectively to

$$\sum_1^3{}_i \left\{ \bar{u}_i (\bar{u}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i (\bar{u}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0 ,$$

$$\sum_1^{\overset{3}{\cancel{4}}}{}_i \left\{ \bar{u}_i (\bar{v}_i - 1) \frac{\partial\varphi}{\partial u_i} + \bar{v}_i (\bar{v}_i - 1) \frac{\partial\varphi}{\partial v_i} \right\} = 0 .$$

Whence the solutions

$$\frac{\bar{u}_1}{\bar{v}_1}, \quad \frac{\bar{u}_2}{\bar{v}_2}, \quad \frac{\bar{u}_3}{\bar{v}_3} .$$

Hence the theorem

If the quadrangle (1234) *is operated on by the transformations of the* LIE *group*

$$\boxed{xq \quad xp - yq \quad yp \quad x^2p + xyq \quad xyp + y^2q}$$

the ratio of the abscissas of any two points varies as the ratio of the ordinates of the two points; the factor of proportionality is different for different pairs.

The reader may find it interesting to construct the other invariants of the quadrangle by the remaining projective subgroups of the plane. He will observe that many of the results admit of immediate extension to a system of n points. Lie has determined all projective groups of the plane; they are tabulated in Chapter XI, § 4, of his Lectures on Continuous Groups.

6. The corresponding theorems for the quadrilateral may be found in the following manner.

An infinitesimal transformation in point coordinates is designated symbolically by

$$Uf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y};$$

by this transformation x and y receive the increments

$$\delta x = \xi(x, y)\,\delta t, \quad \delta y = \eta(x, y)\,\delta t.$$

Let its equivalent transformation in line coordinates be

$$Vf \equiv \varphi(u, v)\frac{\partial f}{\partial u} + \psi(u, v)\frac{\partial f}{\partial v};$$

by which u and v are respectively increased by

$$\delta u = \varphi(u, v)\,\delta t, \quad \delta v = \psi(u, v)\,\delta t.$$

Either of these transformations may be derived from the other by the conditions

$$ux + vy + 1 = 0,$$

$$\delta(ux + vy + 1) = 0.$$

Putting $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $s = \frac{\partial f}{\partial u}$, $t = \frac{\partial f}{\partial v}$, the following table explains its own construction:

$\xi p + \eta q$	$\delta(ux + vy + 1) = 0$	$\varphi s + \psi t$
p	$u + x\delta u + y\delta v = 0$	$u^2 s + uvt$
q	$v + x\delta u + y\delta v = 0$	$uvs + v^2 t$
xp	$ux + x\delta u + y\delta v = 0$	$-us$
xq	$vx + x\delta u + y\delta v = 0$	$-vs$
yp	$uy + x\delta u + y\delta v = 0$	$-ut$
yq	$vy + x\delta u + y\delta v = 0$	$-vt$
$x^2 p + xyq$	$ux^2 + vxy + x\delta u + y\delta v = 0$	s
$xyp + y^2 q$	$uxy + vy^2 + x\delta u + y\delta v = 0$	t

The determinant D, written in (u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4) in place of the x, y, vanishes if the four lines are concurrent, since by the above, the general projective group is seen to have the same form in line coordinates that it has in point coordinates.

The other theorems can be immediately translated by observing that the groups (α), (β), (λ), (μ) in point coordinates assume respectively the forms (β), (α), (μ), (λ) in line coordinates, and the corresponding geometrical interpretations offer no difficulty.

It is perhaps not out of place to call attention to the fact that Lie expresses the property that a group is its own dualistic by enclosing the transformations of the group in a double rectangle thus

$$\boxed{\boxed{p \quad q \quad xp \quad xq \quad yp \quad yq \quad x^2p + xyq \quad xyp + y^2q}}\,.$$

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